October Math Gems

PROBLEM OF THE WEEK 9

§1 Problems

Problem 1.1. When the letters of the alphabet are assigned to their integer values (A = 1, B = 2, C = 3, ..., Z = 26), the word product for JULY is the product of the letter values for each of the letters in JULY. When the square root of JULY's word-product is put into its simplest radical form of $a\sqrt{b}$, where b has no perfect-square factors greater than 1, what is the value of b?

Problem 1.2. In a certain triangle, the size of each of the angles is a whole number of degrees. Also, one angle is $30°$ larger than the average of the other two angles. What is the largest possible angle in this triangle?

Problem 1.3. The points X, Y, and Z are the centers of three circles that touch externally, where each circle touches the other two. The triangle ABC has sides 13, 16, and 20. What are the radii of the three circles?

Problem 1.4. For which values of the positive integer n is it possible to divide the first $3n$ positive integers into three groups each of which has the same sum?

Problem 1.5. what is the unit digit of $1! + 2! + 3! + 4! + ... + 2003!$?

Problem 1.6. A whole number between 1 and 99 is not greater than 90, not less than 30, not a perfect square, not even, not a prime, not divisible by 3, and its last digit is not 5. What is the number?

Problem 1.7. How many triples (x, y, z) of positive integers satisfy $x^{y^z} = 64$?

Problem 1.8. If $2x + 3y + z = 48$ and $4x + 3y + 2z = 69$, what is $6x + 3y + 3z$ equal to?

Problem 1.9. The sum of six consecutive positive odd integers starting with n is a perfect cube. Find the smallest possible n.

Problem 1.10. Let a, b, and c be positive real numbers greater than or equal to 3. Prove that

$$3(abc + b + 2c) \geq 2(ab + 2ac + 3bc)$$

and determine all equality cases

Problem 1.11. Is there a triangle with an area of 12 cm^2 and a perimeter of 12 cm?

Problem 1.12. The real numbers a, b, c and d satisfy simultaneously the equations

$$abc - d = 1, \quad bcd - a = 2, \quad cda - b = 3, \quad dab - c = -6.$$

Prove that $a + b + c + d \neq 0$

Problem 1.13. Let a, b and c be positive real numbers such that $abc = \frac{1}{8}$. Prove the inequality

$$a^2 + b^2 + c^2 + a^2b^2 + b^2c^2 + c^2a^2 \geq \frac{15}{16}$$

When does equality hold?

Problem 1.14. Solve for positive real numbers

$$n + \lfloor \sqrt{n} \rfloor + \lfloor \sqrt[3]{n} \rfloor = 2014$$

Problem 1.15. Let a be a positive real number such that $a^3 = 6(a+1)$. Prove that the equation $x^2 + ax + a^2 - 6$ has no real solutions.

Problem 1.16. Let $a, b, c \geq 0$ and $a + b + c = 2$. Prove that

$$\frac{a}{b+c} + \frac{b}{c+a} + \frac{2c+1}{a+b} \geq \sqrt{10} - \frac{3}{4}$$

Problem 1.17. Let a, b be non-negative real numbers such that $a + b = 2$. Prove that

$$\frac{1}{a^2+1} + \frac{1}{b^2+1} \leq \frac{2}{ab+1}$$

Problem 1.18. Let $x \leq 8$. Prove that

$$x(9-x) + \frac{16}{9-x} \leq 24$$

Problem 1.19. Let a, b, c be positive real numbers such that $ab + bc + ca = 3$. Prove the following inequality

$$\frac{a}{a^2 - bc + 3} + \frac{b}{b^2 - ca + 3} + \frac{c}{c^2 - ab + 3} \leq \frac{3}{a+b+c}.$$

Problem 1.20. Let a, b be non-negative real numbers such that $a + b = 2$. Prove that

$$\frac{a^2+a+1}{b^2-b+3} + \frac{b^2+b+1}{a^2-a+3} \geq 2$$
